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New technique to extend Weak interior ideals to fuzzy setting in Semigroups

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Abstract. Point-wise and set theoretic definitions of fuzzy left(right) weak interior and fuzzy weak interior ideals within a semigroup are presented. We further explores the characteristics of fuzzy left weak(right weak, weak)interior ideals, examining their relationships with fuzzy left (right, two-sided ideals, bi-, interior, quasi-)ideals, via utilizing Tom Head's metatheorem. The characterization of fuzzy weak interior ideals is discussed in terms of both level subsets and strong level subsets. Additionally, it is demonstrated that the classes of fuzzy weak interior ideals, along with other types of fuzzy ideals in a semigroup, exhibit projection closed property. The metatheorem is employed to derive fuzzy analogues of classical results, enabling proofs for several propositions related to fuzzy weak interior ideals in a semigroup without the need for extensive calculations. These proofs, facilitated by the metatheorem, are notably concise, straightforward, and devoid of complex computations. Furthermore, the paper provides characterizations of regular semigroups based on fuzzy weak interior ideals.

Keywords. Semigroups; fuzzy weak interior ideals; metatheorem; projection closed; Rep function.

AMS Subject Classification Number-20M12

1 Introduction

L. A. Zadeh [19] in 1965 introduced the notion of fuzzy set to describe the vagueness mathematically. Rosenfeld [16] introduced concepts of fuzzy subgroupoids and fuzzy subgroups. Kumar, R. et.al. [1,2] and Khumbhojkar [6,7] investigated different types of fuzzy ideals in semigroups and rings.

The concept of bi-ideals was initially introduced by Lajos and Szasz [11] in the context of associative rings, and subsequently expanded by Good and Hughes [3] to semigroups. Szasz [18] introduced the notion of interior ideals in semigroups, while Kuroki [8-10] further advanced the theory by defining fuzzy bi-ideals and fuzzy interior ideals within semigroups. The generalization of ideals within algebraic structures is not only a significant area of study for mathematicians but is also essential for advancing the understanding of these structures. Between 1950 and 1980, numerous researchers focused on the study of (bi-, quasi-, interior) ideals. One-sided ideals provide a broader framework for ideals, while quasi-ideals further extend this framework to encompass both left and right ideals. Bi-ideals represent a generalization of quasi-ideals. Between 2020 and 2022, Rao [13,14,15] introduced the notion of weak interior ideals as a generalization of left, right, and two-sided ideals, as well as quasi-ideals and interior ideals, in both semigroups and semirings.

In this work, we begin with both set theoretic and point-wise definitions of fuzzy left weak(right weak, weak)interior ideals in a semigroup. Later we examine fuzzy left weak(right weak, weak)interior ideals in semigroups, which extend the concepts of fuzzy left(right, two-sided ideals, interior ideals, quasi-)ideals. The properties of fuzzy weak interior ideals are explored through the application of metatheorem formulated by Tom Head [4], which is also used to characterize these ideals in both semigroups and simple semigroups. The metatheorem offers techniques for extending classical algebraic results to the fuzzy context. This approach was later adopted in fuzzy algebra by [5,17], underscoring its clarity and conciseness compared to existing methods, and demonstrating its utility as an effective tool for studying fuzzy algebraic structures.

2 Preliminaries

Definition 2.1. [13] A non-empty subset A of a semigroup S is said to be left (right) weak-interior ideal of S if A is a subsemigroup of S and $SAA \subseteq A$ ($AAS \subseteq A$) and is called a weak interior ideal if A is a subsemigroup of S and is a left and a right weak-interior ideal of S.

We recall a *fuzzy set* μ on a non-set X defined by Zadeh[19] as a mapping $\mu : X \to [0,1]$. Throughout this paper, J will always denote $[0,1[$ and also C_{ss}, $C_l(C_r, C_i, C_b, C_{in}, C_q, C_{wil}, C_{wir}, C_{wi})$ denotes the class of crisp subsemigroups, left (right, two sided, bi-, interior, quasi-, left weak interior, right weak interior, weak interior) ideals of an ordered semigroup S. Also the class $\mathcal{C}_{ss}$, $\mathcal{C}_l(\mathcal{C}_r, \mathcal{C}_i, \mathcal{C}_b, \mathcal{C}_{in}, \mathcal{C}_q, \mathcal{C}_{wil}, \mathcal{C}_{wir}, \mathcal{C}_{wi})$ denotes their corresponding fuzzy classes respectively.

Definition 2.2. [8, 16] A fuzzy set μ of S is said to be a

(1) *fuzzy subsemigroup* of S if $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ $\forall$ $x, y \in S$.

(2) *fuzzy left(right)* of S if $\mu(xy) \geq \mu(y)$ $(\mu(xy) \geq \mu(x))$, $\forall$ $x, y \in S$.

(3) *fuzzy ideal* of S $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$ $\forall$ $x, y \in S$.

Definition 2.3. [9, 10] $\mu \in \mathcal{C}_{ss}$ is set to be a

(1) *fuzzy bi-ideal* of S if $\mu(xyz) \geq \min\{\mu(x), \mu(z)\}\ \forall\ x, y, z \in S$.

(2) *fuzzy interior ideal* of S if $\mu(xyz) \geq \mu(y)\ \forall\ x, y \in S$.

Definition 2.4. [17] $\mu \in F(S)$ is said to be a *fuzzy quasi-ideal* of S if $f(xy) \geq \min\{f(x), f(y)\}\quad x, y \in S$ and $\mu(z) \geq \min\{\sup\limits_{z=xy} \mu(x), \sup\limits_{z=xy} \mu(y)\}\ \forall$ $z \in S$.

Definition 2.5. $\mu \in \mathcal{C}_{ss}$ is said to be a *fuzzy left(right) weak interior ideal* of S if $\mu \circ \mu \circ S \subseteq \mu(S \circ \mu \circ \mu \subseteq \mu)$

Definition 2.6. $\mu \in \mathcal{C}_{ss}$ is said to be a *fuzzy weak interior ideal* of S if $S \circ \mu \circ \mu \subseteq \mu$ and $\mu \circ \mu \circ S \subseteq \mu$.

A briefly description of 'metatheorem' derived by Tom Head [4] in the field of fuzzy algebra presented in this section. $P(S)$ and $C(S)$ denotes the set of all subset and characteristic functions of S. The characteristic function $Chi : P(S) \to C(S)$, defined as $Chi(A) = \chi_A$, is one-one and onto.

Proposition 2.7. *$P(S) \cong C(S)$ under the isomorphism Chi.*

Proposition 2.8. *The characteristic function Chi commutes with both the finite intersection and the finite product of sets within a semiring S.*

Definition 2.9. [4] For $\mu \in F(S)$ and $r \in J = [0, 1[$, the function $\mathrm{Rep} : F(S) \to C(S)^J$ is defined by

$$\mathrm{Rep}(\mu)(r)(x) = \begin{cases} 1 & \text{if } \mu(x) > r \\ 0 & \text{if } \mu(x) \leq r \end{cases}$$

Proposition 2.10. [4] $\mathrm{Rep}(\bigcap\limits_{i=1}^{k} \mu_i) = \bigcap\limits_{i=1}^{k} \mathrm{Rep}(\mu_i)$ *and* $\mathrm{Rep}(\bigcup\limits_{i=1}^{\infty} \mu_i) = \bigcup\limits_{i=1}^{\infty} \mathrm{Rep}(\mu_i)$, *where $\mu_i \in F(S)$.*

Proposition 2.11. [4] *The function* Rep *constitutes an order isomorphism from* $F(S)$ *onto* $I(S)$*, where* $I(S)$ *denotes the image of function* Rep.

We define a binary operation $* : F(S) \times F(S) to F(S)$ as follows:

$$(\mu_1 * \mu_2)(x) = \begin{cases} \sup\limits_{x=x_1*x_2} [\min\{\mu_1(x_1), \mu_2(x_2)\}] \\ 0, \qquad\qquad \text{if } x \text{ not expressed as } x = x_1 * x_2 \end{cases}$$

Tom Head [4] introduced the binary operation $*$ on F(S) as a convolutional extension of the binary operation $*$ on S. Notably, this convolutional extension on $F(S)$ corresponds to the fuzzy set product proposed by Liu [12], which is applicable within a semigroup or any other algebraic structure.

Proposition 2.12. [4] *For* $A, B \in P(S)$*,* $Chi(A * B) = ChiA * ChiB$.

Proposition 2.13. [4] *For* $\mu_1, \mu_2 \in F(S)$*,* $\mathrm{Rep}(\mu_1 * \mu_2) = \mathrm{Rep}(\mu_1) * \mathrm{Rep}(\mu_2)$.

Definition 2.14. [4] A class $\mathcal{C}$ of fuzzy sets within a semigroup S is projection closed if, for each $\mu \in \mathcal{C}$ and $r \in J$, $Rep(\mu)(r) \in \mathcal{C}$.

Proposition 2.15. [4] *Let* $C_1, C_2(\mathcal{C}_1, \mathcal{C}_2)$ *be the classes of crisp (fuzzy) subsets of a semigroup* S*. Then,* $\mathcal{C}_1 \subseteq \mathcal{C}_2(\mathcal{C}_1 = \mathcal{C}_2) \Leftrightarrow C_1 \subseteq C_2(C_1 = C_2)$.

Metatheorem 2.16. [4] *Consider a semigroup* $(S, *)$ *and the algebra* $(F(S), inf, sup, *)$*. Let* $L(a_1, a_2, \ldots, a_m)$ *and* $M(a_1, a_2, \ldots, a_m)$ *are expression over the variables set* $\{a_1, a_2, \ldots, a_m\}$ *and operations set* $\{\inf, \sup, *\}$ *defined on* $P(S)$*. Let* $\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_m$ *are projection closed classes of fuzzy sets of* S *and* $D_1, D_2, \ldots, D_m$ *be their corresponding crisp classes. Then the inequality*

$$L(\mu_1, \mu_2, \ldots, \mu_m) \quad \mathrm{REL} \quad M(\mu_1, \mu_2, \ldots, \mu_m)$$

hold $\forall\ \mu_1 \in \mathcal{D}_1, \ldots, \mu_m$ *in* $\mathcal{D}_m \Leftrightarrow$ *it holds* $\forall\ \mu_1$ *in* $D_1, \ldots, \mu_m$ *in* D_m *where* REL *symbolizes any of the three specific relations* $\leq$, $=$ *or* $\geq$.

3 Fuzzy weak interior ideals in semigroups

Definition 3.1. $\mu \in \mathcal{C}_{ss}$ is called a *fuzzy left weak interior ideal* of S if $\mu(w) \geq \sup_{w=xyz} \min[\mu(y), \mu(z)] \ \forall\ w \in S$.

Now we establish the equivalence between the Definition 2.5 and 3.1.

Theorem 3.2. $\mu \in \mathcal{C}_{wil}$ *if and only if* $S \circ \mu \circ \mu \subseteq \mu$.

Proof. Consider $z \in S$. Then, $S \circ \mu \circ \mu \subseteq \mu$

$\Leftrightarrow (S \circ \mu \circ \mu)(z) \leq \mu(z)$

$\Leftrightarrow \sup_{z=xyw} [\min(S \circ \mu)(xy), \mu(w))] \leq \mu(z)$

$\Leftrightarrow \sup_{z=xyw} \min[\min\{S(x), \mu(y)\}, \mu(w)] \leq \mu(z)$

$\Leftrightarrow \sup_{z=xyw} \min[\mu(y), \mu(w)] \leq \mu(z)$

$\Leftrightarrow \mu \in \mathcal{C}_{wil}$. □

Definition 3.3. $\mu \in \mathcal{C}_{ss}$ is said to be a *fuzzy right weak-interior ideal* of S if $\mu(w) \geq \sup_{w=xyz} \min[\mu(x), \mu(y)] \ \forall\ w \in S$.

Next we establish the equivalence between the Definition 2.5 and 3.3.

Theorem 3.4. $\mu \in \mathcal{C}_{wir}$ *if and only if* $\mu \circ \mu \circ S \subseteq \mu$.

Proof. Let $z \in S$. Then, $\mu \circ \mu \circ S \subseteq \mu$

$\Leftrightarrow (\mu \circ \mu \circ S)(z) \leq \mu(z)$

$\Leftrightarrow \sup_{z=xyw} [\min(\mu \circ \mu)(xy), S(w))] \leq \mu(z)$

$\Leftrightarrow \sup_{z=xyw} \min[\min\{\mu(x), \mu(y)\}, S(w)] \leq \mu(z)$

$\Leftrightarrow \sup_{z=xyw} \min[\mu(x), \mu(y)] \leq f(z)$

$\Leftrightarrow \mu \in \mathcal{C}_{wir}$. □

Definition 3.5. $\mu \in \mathcal{C}_{ss}$ is said to be a *fuzzy weak interior ideal* of S if $\sup_{z=xyw} \min[\mu(y), \mu(w)] \leq \mu(z)$ and $\sup_{z=xyw} \min[\mu(x), \mu(y)] \leq \mu(z)$.

Finally the equivalence between Definition 2.5 and 3.5 can be established in a similar manner.

Theorem 3.6. *$\mu \in \mathcal{C}_{wi} \Leftrightarrow S \circ \mu \circ \mu \subseteq \mu$ and $\mu \circ \mu \circ S \subseteq \mu$.*

4 Level subset and strong level subsets in semigroups

Definition 4.1. For $\mu \in F(X)$ and $r \in [0, 1[$, the level subset μ_t and strong level subset $\mu_t^>$ associated with μ are described as follows:

$$\mu_t = \{x \in X : \mu(x) \geq r\} \quad \text{and} \quad \mu_t^> = x \in X : \mu(x) > r\}$$

Lemma 4.2. *For $\gamma, \mu \in F(S)$. Then, $(\gamma \circ \mu)_t^> = \gamma_t^> \mu_t^> \ \forall\ t \in [0, 1[$.*

Theorem 4.3. *The intersection of any non-empty collection of fuzzy left weak interior ideals in a semigroup S remains a fuzzy left weak interior ideal.*

Proof. Consider $\{\mu_i\}_{i\in\delta}$ as a collection of fuzzy left weak interior ideals in the semigroup S and let $\mu = \bigcap\limits_{i\in\delta} \mu_i$, where δ represents an indexing set. Since the intersection of any non-empty collection of fuzzy subsemigroups results in a semigroup that is also a fuzzy subsemigroup implies that μ comprises a fuzzy subsemigroup within S. For any $x, y, z \in S$, $\mu(xyz) = \bigcap\limits_{i\in\delta} \mu_i(xyz) = \inf\limits_{i\in\delta} \mu_i(xyz)$ $\geq \inf\limits_{i\in\delta} \min\{\mu_i(y), \mu_i(z)\} = \min\{\inf\limits_{i\in\delta} \mu_i(y), \inf\limits_{i\in\delta} \mu_i(z)\} = \min\{\bigcap\limits_{i\in\delta} \mu_i(y), \bigcap\limits_{i\in\delta} \mu_i(z)\}$. Thus $\mu \in \mathcal{C}_{wil}$. □

Theorem 4.4. *The intersection of any non-empty collection of fuzzy right (weak interior, weak interior) ideals in a semigroup S remains a fuzzy right (weak interior, weak interior) ideal.*

Theorem 4.5. *The following assertions are mutually equivalent in a semigroup S:*

(1) $\mu \in \mathcal{C}_{wil}$.

(2) *Each* $\emptyset \neq \mu_t$ *is a left weak interior ideal of* S.

(3) *Each* $\emptyset \neq \mu_t^{>}$ *is a left weak interior ideal of* S.

Proof. To establish $(\mathbf{1}) \Rightarrow (\mathbf{3})$, consider $\mu \in \mathcal{C}_{wil}$. We claim that $S\mu_t^{>}\mu_t^{>} \subseteq \mu_t^{>}$. Let $w \in S\mu_t^{>}\mu_t^{>}$. Then $w = xyz$ for some $x \in S$ and $y, z \in \mu_t^{>}$. Then $\mu(y) > t < \mu(z)$ as $y, z \in \mu_t$. Since $\mu \in \mathcal{C}_{wil}$, therefore, $\mu(xyz) \geq \min\{\mu(y), \gamma(z)\} > t$. Thus $w = xyz \in \gamma_t^{>}$. Hence $S\gamma_t^{>}\gamma_t^{>} \subseteq \gamma_t^{>}$.

$(\mathbf{3}) \Rightarrow (\mathbf{2})$. On contrary consider $\emptyset \neq \mu_t$, a level subset of μ. Then, $\mu_t = \bigcap\limits_{r<t} \mu_t^{>}$. Since $\emptyset \neq \mu_t$, $\emptyset \neq \mu_t^{>}$ for each $r < t$. By (3) $\mu_t^{>}$ is a left weak interior ideal of S. Since the intersection of any non-empty set of left weak interior ideals in S results in a structure that is also a left weak interior ideal of S, it follows that μ_t is a left weak interior ideal.

$(\mathbf{2}) \Rightarrow (\mathbf{1})$. Suppose $\mu \notin \mathcal{C}_{wil}$. Since μ_t is a left weak interior ideal of S, it is a subsemigroup and thus $\mu \in \mathcal{C}_{ss}$. Since $\mu \notin \mathcal{C}_{wil}$, $\mu(xyz) < \min\{\{\mu(y), \mu(z)\}$ for some $x, y, z \in S$. This implies $\mu(a) < \min\{\min\{\mu(y), \mu(z)\}$ where a=xyz. Select a real number t such that $\mu(a) < t < \min\{\mu(y), \mu(z)\}$. Therefore, $a \notin \mu_t$ and $y, z \in \mu_t$. Therefore, $a = xyz \in S\mu_t\mu_t$, but $a \notin \mu_t$. Hence $S\mu_t\mu_t \nsubseteq \mu_t$, which contradicts (2). □

Similarly we can prove:

Theorem 4.6. *In a semigroup* S, *the following statements are mutually equivalent:*

(1) $\mu \in \mathcal{C}_{wir}$.

(2) *Each* $\emptyset \neq \mu_t$ *is a right weak interior (weak interior) ideal of* S.

(3) *Each* $\emptyset \neq \mu_t^{>}$ *is a right weak interior (weak interior) ideal of* S.

5 Projection closed fuzzy classes in Semigroup

Theorem 5.1. [13] *$\mathcal{C}_{ss}$, $\mathcal{C}_l$, $\mathcal{C}_r$, $\mathcal{C}_i$, $\mathcal{C}_{in}$, $\mathcal{C}_b$ and $\mathcal{C}_q$ are projection closed.*

Theorem 5.2. *$\mathcal{C}_{wil}$ is projection closed.*

Proof. Consider $\mu \in \mathcal{C}_{wil}$. Then $\mu \in \mathcal{C}_{ss}$ and $\mu(xyz) \geq \min\{\mu(y), \mu(z)\}\ \forall\ x, y, z \in S$. In light of Theorem 5.1, it is adequate to demonstrate that $\text{Rep}(\mu)(r)(xyz) \geq \min\{\text{Rep}(\mu)(r)(y), \text{Rep}(\mu)(r)(z)\}\ \forall\ x, y, z \in S$ and $\forall\ r \in J\ \forall$. Let $\min\{\text{Rep}(\mu)(r)(y), \text{Rep}(\mu)(r)(z)\} = 1$. Then, $\text{Rep}(\mu)(r)(y)$= $\text{Rep}(\mu)(r)(z)$=1. Therefore, $\mu(y) > r < \mu(z)$. Therefore, $\mu(xy) \geq \min\{(\mu(y), \mu(z)\} > r$. Thus $\text{Rep}(\mu)(r)(xyz) = 1$. If $\min\{\text{Rep}(\mu)(r)(y), \text{Rep}(\mu)(r)(z)\} = 0$, then the inequality holds trivially. Hence $\mathcal{C}_{wil}$ is projection closed. □

Theorem 5.3. *$\mathcal{C}_{wir}$ is projection closed.*

Proof. Consider $\mu \in \mathcal{C}_{wir}$. Then $\mu \in \mathcal{C}_{ss}$ and $\mu(xyz) \geq \min\{\mu(x), \mu(y)\}\ \forall\ x, y, z \in S$. In light of Theorem 5.1, it is adequate to demonstrate that $\text{Rep}(\mu)(r)(xyz) \geq \min\{\text{Rep}(\mu)(r)(x), \text{Rep}(\mu)(r)(y)\}\ \forall\ x, y, z \in S$ and $\forall\ r \in J\ \forall$. Let $\min\{\text{Rep}(\mu)(r)(x), \text{Rep}(\mu)(r)(y)\} = 1$. Then, $\text{Rep}(\mu)(r)(x)$= $\text{Rep}(\mu)(r)(y)$=1. Therefore,$\mu(x) > r < \mu(y)$. Therefore, $\mu(xyz) \geq \min\{(\mu(x), \mu(y)\} > r$. Thus $\text{Rep}(\mu)(r)(xyz) = 1$. If $\min\{\text{Rep}(\mu)(r)(x), \text{Rep}(\mu)(r)(y)\} = 0$, then the inequality holds trivially. Hence $\mathcal{C}_{wir}$ is projection closed. □

Theorem 5.4. *$\mathcal{C}_{wi}$ is projection closed.*

6 Fuzzy weak-interior ideals in semigroups

Theorem 6.1. *For a semigroup S, we have,*

(1) *Each fuzzy left (or right) ideal of S is also a fuzzy left (or right) weak-interior ideal.*

(2) *Each fuzzy ideal of S is also a fuzzy weak-interior ideal of the semigroup.*

(3) *Each fuzzy quasi ideal of S is also a fuzzy weak-interior ideal of the semigroup.*

(4) *Each fuzzy interior ideal of S is also a fuzzy left(right) weak-interior ideal of the semigroup of S.*

Proof. We demonstrate (3). Since both the classes $\mathcal{C}_q$ and $\mathcal{C}_{wi}$ are projection closed, therefore, $\mathcal{C}_q \subseteq \mathcal{C}_{wi} \Leftrightarrow C_q \subseteq C_{wi}$ by Proposition 2.15. By Corollary 3.8 of [13], every quasi-ideal of a semigroup S is a weak-interior ideal and $P(S) \cong C(S)$ under the isomorphism Chi, we get, $C_q \subseteq C_{wi}$. Hence $\mathcal{C}_q \subseteq \mathcal{C}_{wi}$.

The remaining parts, which extends Theorem 4.5 and 4.6 of [13] to fuzzy framework and can be derived analogously. □

The following theorem is an extension of Theorem 4.17 of [13] to fuzzy setting and can be prove similarly.

Theorem 6.2. *In a left simple semigroup S, every fuzzy left weak-interior ideal is also a fuzzy two-sided ideal within S.*

Theorem 6.3. *In a semigroup S, the following statements hold:*

(1) *If $\gamma \in \mathcal{C}_{wi}$ and $\mu \in \mathcal{C}_{ss}$, then $\gamma \cap \mu \in \mathcal{C}_{wi}$.*

(2) *If $\gamma \in \mathcal{C}_{wil}$ and $\mu \in \mathcal{C}_r$, then $\gamma \cap \mu \in \mathcal{C}_{wil}$.*

Proof. To establish (1), define the classes $C = \{\gamma \cap \chi_A \ : \ A \in P(S)$ and $A^2 \subset A$ and $\gamma \in C_{wi}\}$ and $\mathcal{C} = \{\gamma \cap \mu : \gamma \in \mathcal{C}_{wi}$ and $\mu \in \mathcal{C}_{ss}\}$ in S. To begin, we demonstrate that $\mathcal{C}$ is projection closed. Let $\gamma \cap \mu \in \mathcal{C}$. Now, $\mathrm{Rep}(\gamma \cap \mu)(r) = \mathrm{Rep}(\gamma)(r) \cap \mathrm{Rep}(\mu)(r)\ \forall\ r \in J$ by Proposition 2.10. Since both

$\mathcal{C}_{ss}$ and $\mathcal{C}_{wi}$ are projection closed, therefore, $\mathrm{Rep}(\gamma)(r) \in \mathcal{C}_{wi}$ and $\mathrm{Rep}(\mu)(r) \in \mathcal{C}_{ss}$. Thus $\forall\ r \in J$, $\mathrm{Rep}(\gamma \cap \mu)(r) \in \mathcal{C}$ and hence $\mathcal{C}$ is closed under projection. By Theorem 4.5 of [13], the product of a weak interior-ideal and a subsemigroup is a weak interior ideal and $P(S) \cong C(S)$ under the isomorphism Chi, we get, $C \subseteq C_{wi}$. Hence $\mathcal{C} \subseteq \mathcal{C}_{wi}$ by Proposition 2.15. □

Theorem 6.4. *Let $\gamma \in \mathcal{C}_l$ and $\mu \in \mathcal{C}_r$, then $\gamma \circ \mu \in \mathcal{C}_{wi}$*

Proof. Consider classes $C_{r,l} = \{f_1 \circ f_2\colon\ f_1 \in C_r,\ f_2 \in C_l\}$ and $\mathcal{C}_{r,l} = \{\gamma \circ \mu\colon \gamma \in \mathcal{C}_r,\ \mu \in \mathcal{C}_l\}$ in S. To begin, we demonstrate that $\mathcal{C}_{r,l}$ is projection closed. Let $\gamma \circ \mu \in \mathcal{C}_{r,l}$. For all $\gamma \in \mathcal{C}_r$ and $\mu \in \mathcal{C}_l$, $\mathrm{Rep}(\gamma \circ g)(r) = \mathrm{Rep}(\gamma)(r) \circ \mathrm{Rep}(\mu)(r)\ \forall\ r \in J$ by Proposition 2.13. Since both $\mathcal{C}_r$ and $\mathcal{C}_l$ are projection closed, we have, $\mathrm{Rep}(\gamma)(r) \in \mathcal{C}_r$ and $\mathrm{Rep}(\mu)(r) \in \mathcal{C}_l$. Thus, $\mathrm{Rep}(\gamma \circ \mu)(r) \in \mathcal{C}_{r,l}\ \forall\ r \in J$. Hence, $\mathcal{C}_{r,l}$ is projection closed. Also $\mathcal{C}_{wi}$ is projection closed. Since the product of a left ideal and a right ideal of a semigroup is a weak interior ideal by Theorem 4.5 of [13] and $P(S) \cong C(S)$ under the isomorphism Chi, we get, $C_{r,l} \subseteq C_{wi}$. Hence $\mathcal{C}_{r,1} \subseteq \mathcal{C}_{wi}$ by Proposition 2.15. □

Theorem 6.5. *Let $h \in \mathcal{C}_{ss}$. Then $h \in \mathcal{C}_{wil}$ if and only if there exists $\mu \in \mathcal{C}_l$ such that $\mu \circ \mu \subseteq h \subseteq \mu$.*

Proof. Consider classes $C = \{\chi_H \in C(S)$, where H is a subsemigroup of S: $f_1 f_1 \subseteq H \subseteq f_1$ for some left ideal $f_1\ of\ S\}$ and $\mathcal{C} = \{h_1 \in \mathcal{C}_{ss}\colon\ \mu \circ \mu \subseteq h \subseteq \mu$ for some $\mu \in \mathcal{C}_l\}$ in S. To begin we demonstrate that $\mathcal{C}$ is projection closed. Consider $h \in \mathcal{C}$, therefore, $h \in \mathcal{C}_{ss}$ such that $\mu \circ \mu \subseteq h \subseteq \mu$ for some $\mu \in \mathcal{C}_l$. For $r \in J$, $\mathrm{Rep}(\mu \circ \mu)(r) \geq \mathrm{Rep}(h)(r) \geq \mathrm{Rep}(\mu)(r)$ by Proposition 2.11. By Proposition 2.13, for $r \in J$, $\mathrm{Rep}(\mu) \circ \mathrm{Rep}(\mu)(r) \geq \mathrm{Rep}(h)(r) \geq \mathrm{Rep}(\mu)(r)$. Since $\mathcal{C}_{ss}$ and $\mathcal{C}_l$ are projection closed, we get, $\mathrm{Rep}(\mu)(r) \in \mathcal{C}_l$ and $\mathrm{Rep}(h)(r) \in \mathcal{C}_{ss}$. Therefore, $\mathrm{Rep}(\mu)(r) \in \mathcal{C}$ for some $\mu \in \mathcal{C}_l$ and $\forall\ r \in J$ and hence $\mathcal{C}$ is projection closed. Also the class $\mathcal{C}_{wil}$ is projection closed. Therefore, $\mathcal{C} \subseteq \mathcal{C}_{wil}$ if and only if $C \subseteq C_{wil}$ by Proposition 2.15.

The later proposition follows as subsemigroup K in a semigroup S is a left weak interior ideal if and only if $A^2 \subseteq K \subseteq A$ for some left ideal A of S by Theorem 4.7 of [13] and $P(S) \cong C(S)$ under the isomorphism Chi. $\square$

Theorem 6.6. *Let $h \in \mathcal{C}_{ss}$. Then $h \in \mathcal{C}_{wir}$ if and only if there exists $\mu \in \mathcal{C}_r$ such that $\mu \circ \mu \subseteq h \subseteq \mu$.*

Theorem 6.7. *In a semigroup S that is idemempotent and regular, $\mu \in \mathcal{C}_{wi} \Leftrightarrow \mu \circ \mu \circ S = \mu = S \circ \mu \circ \mu \ \forall \ \mu \in \mathcal{C}_{wil}$.*

Proof. We define the classes $D = \{H \in C_{wi} : HHS = H = SHH\}$ and $\mathcal{D} = \{\mu \in \mathcal{C}_{wi} : \mu \circ \mu \circ S = \mu = S \circ \mu \circ \mu\}$. Firstly we show that $\mathcal{D}$ is projection closed. Let $\mu \in \mathcal{D}$. Therefore, $\mu \in \mathcal{C}_{wi}$ such that $\mu \circ \mu \circ S = \mu = S \circ \mu \circ \mu$. $\mathrm{Rep}(\mu \circ \mu \circ S)(r) = \mathrm{Rep}(\mu)(r) = \mathrm{Rep}(S \circ \mu \circ \mu)(r) \ \forall \ r \in J$ by Proposition 2.11. We have, for all $\mu \in \mathcal{C}_{wi}$, $\mathrm{Rep}(\mu)(r) \circ \mathrm{Rep}(\mu)(r) \circ \mathrm{Rep}(S)(r) = \mathrm{Rep}(\mu)(r) = \mathrm{Rep}(S)(r) \circ \mathrm{Rep}(\mu)(r) \circ \mathrm{Rep}(\mu)(r) \ \forall \ \ r \in J$ by Proposition 2.13. That is, for all $\mu \in \mathcal{C}_{wi}$, $\mathrm{Rep}(\mu)(r) \circ \mathrm{Rep}(\mu)(r) \circ S = \mathrm{Rep}(\mu)(r) = S \circ \mathrm{Rep}(\mu)(r) \circ \mathrm{Rep}(\mu)(r)$ $\forall \ r \in J$. Since $\mathcal{C}_{wi}$ is projection closed, we have, $\mathrm{Rep}(\mu)(r) \in \mathcal{C}_{wi}$. Hence $\mathrm{Rep}(f)(r) \in \mathcal{D} \ \forall \ r \in J$. Thus, $\mathcal{D}$ is projection closed. Also $\mathcal{C}_{wi}$ is projection closed. Therefore, $\mathcal{D} = \mathcal{C}_{wi} \Leftrightarrow D = C_{wi}$ By Theorem 2.15. Since a subsemigroup A_1 of a semigroup S that is idempotent and regular is a weak interior ideal if and only if $A_1A_1S = A_1 = SA_1A_1$ by Theorem 4.21 of [13] and $P(S) \cong C(S)$ under the isomorphism Chi, we get, $D = C_{wi}$. Thus $\mathcal{D} = \mathcal{C}_{wi}$. $\square$

Theorem 6.8. *Consider $\mu \in \mathcal{C}_{ss}$. Then $\mu \in \mathcal{C}_{wi}$ if $S \circ S \circ \mu \subseteq \mu$ and $\mu \circ S \circ S \subseteq \mu$.*

Proof. Consider $C = \{\chi_H \in C(S)$, where H is a subsemigroup of S: $SSH \subseteq H$ and $HSS \subseteq H\}$ and $\mathcal{C} = \{\mu \in \mathcal{C}_{ss} : S \circ S \circ \mu \subseteq \mu$ and $\mu \circ S \circ S \subseteq \mu\}$ as fuzzy and crisp classes in S. Utilizing Proposition 2.10, 2.11 and proceeding as in Theorem 6.7, it can be shown easily that $\mathcal{C}$ is projection closed. Also by Theorem 5.4, $\mathcal{C}_{wi}$ is projection closed. Therefore, $\mathcal{C} \subseteq \mathcal{C}_{wi} \Leftrightarrow C \subseteq C_{wi}$ by Proposition 2.15, here C_{wi} is a class of all crisp weak interior-ideals of S. Since a subsemigroup A_1 of a semigroup S is a weak interior ideal if $SSA_1 \subseteq A_1$ and $A_1SS \subseteq A_1$ by Theorem 4.6 of [13] and $P(S) \cong C(S)$ under the isomorphism Chi, we get, $C \subseteq C_{wi}$. Hence $\mathcal{C} \subseteq \mathcal{C}_{wi}$. $\square$

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